

A guide for teachers - Years 11 and 12

Algebra and coordinate geometry: Module 1

Algebra review



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Algebra review - A guide for teachers (Years 11-12)

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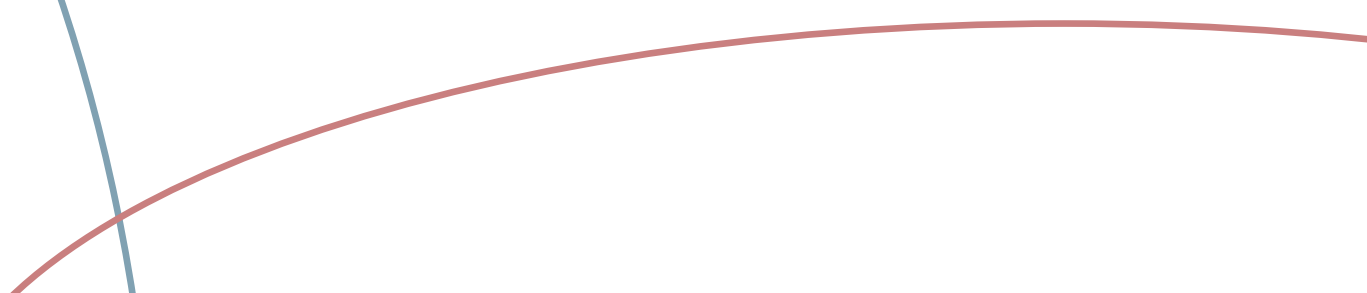
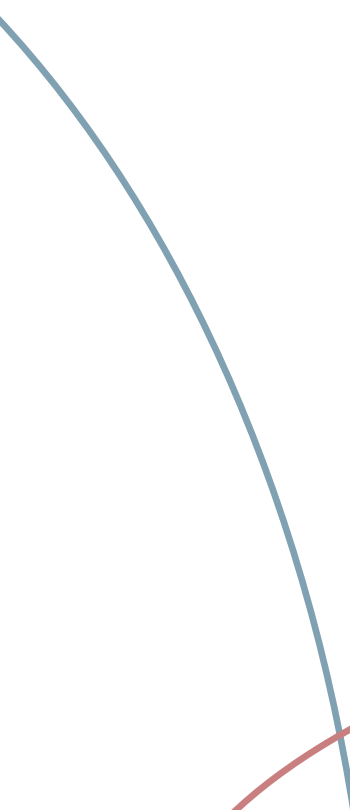
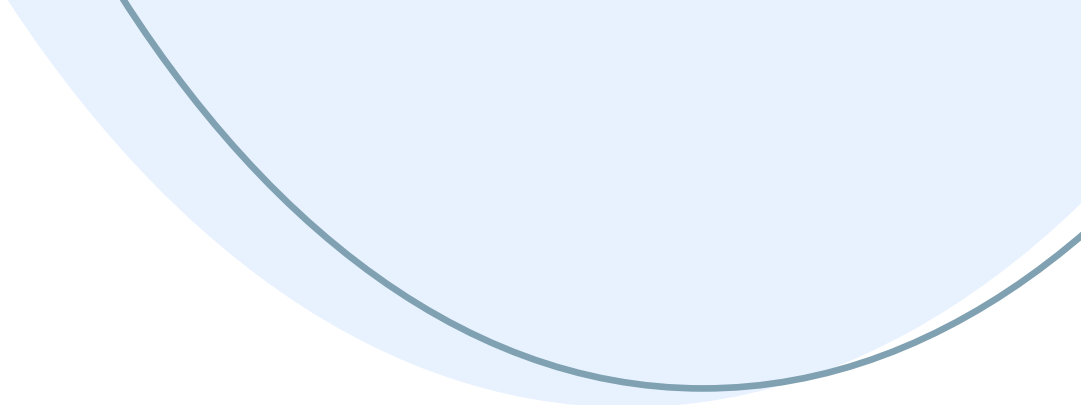
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Algebra review

Assumed knowledge

- Basic skills for simplifying algebraic expressions and algebraic fractions.
- Expanding brackets.
- Factorising linear and quadratic expressions.
- Solving linear equations and inequalities.
- Solving quadratic equations by factoring, completing the square and applying the quadratic formula.
- Solving simultaneous linear equations.
- Substitution.

Motivation

Algebra is the language of mathematics. Just as ordinary language is essential in verbal and written communication, so the ability to read, write and simplify algebraic expressions is fundamental to mathematical understanding, problem solving and the communication of mathematical ideas.

Students need to be able to write mathematics succinctly and correctly. The equal sign must only be placed between two quantities or expressions that are equal and never between two that are not. The solution to a problem should be able to be read so that each line follows from the previous line in a way that makes sense logically and symbolically. This is a skill that takes students some time and effort to master and it is the teacher's role here to act as a model *par excellence* and as a corrector of poorly written work.

A failure to write mathematics logically and correctly leads to and reflects a failure to think logically.

All of the algebra covered in this review is covered in much greater detail in the TIMES modules (www.amsi.org.au/school-education/times-modules).

The relevant TIMES modules are:

- *Algebraic expressions* (Year 7)
- *Linear equations* (Years 7–8)
- *Negatives and the index laws in algebra* (Years 7–8)
- *Fractions and the index laws in algebra* (Years 8–9)
- *Special expansions and algebraic fractions* (Years 8–9)
- *Factorisation* (Years 9–10)
- *Quadratic equations* (Years 9–10)
- *Indices and logarithms* (Years 9–10).

Students and teachers should be giving some thought as to the logical flow when solving equations and writing algebra. When we write

$$x - 3 = 5, \quad x = 8,$$

we are really saying that each of these statements is logically equivalent to the other. We really mean, of course, that $x - 3 = 5$ implies $x = 8$ and $x = 8$ implies $x - 3 = 5$. This can be written formally as $x - 3 = 5 \iff x = 8$, but we do not use this sort of notation in school mathematics. On the other hand, when we write

$$x = 3, \quad \text{so } x^2 = 9,$$

we need to be aware that the implication only goes one way. That is, the statement $x^2 = 9$ does not necessarily imply that $x = 3$.

This module will provide a summary and review of this material. It is left up to the teacher to decide how much of this needs to be retaught and how much time should be spent on it. A degree of comfort and fluency with the content of this module is, however, essential before tackling calculus.

Content

Algebraic manipulation

Simplification

An **algebraic expression** consists of pronumerals, numbers and operations. Only *like terms* can be added or subtracted.

Example

Simplify

$$3x^2 - 4xy + 3x - 7 - 5y + 2yx + 12x^2 - 8x + 7.$$

Solution

$$3x^2 - 4xy + 3x - 7 - 5y + 2yx + 12x^2 - 8x + 7 = 15x^2 - 2xy - 5x - 5y.$$

Note that $xy = yx$ and that it is usual, but not necessary, to write the terms in order of powers, often from highest to lowest.

The index laws

Indices are initially a convenient way to represent repeated multiplication. Once this is done, we can state a number of obvious laws that arise naturally to assist in simplification. The meaning of the indices can then be extended to include integers and rational numbers.

A fluency and comfort with the use of indices is essential before dealing with differential calculus, whose basic rules involve the manipulation of indices.

The index laws were covered in the TIMES modules:

- *Negatives and the index laws in algebra* (Years 7–8)
- *Fractions and the index laws in algebra* (Years 8–9)
- *Indices and logarithms* (Years 9–10).

The index laws state that, for a, b non-zero and m, n positive integers,

$$a^m a^n = a^{m+n} \qquad a^m \div a^n = a^{m-n}$$

$$(ab)^n = a^n b^n \qquad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

$$(a^m)^n = a^{mn}.$$

These laws hold for any non-zero real numbers a and b . The laws also hold when m, n are negative integers and rational numbers. Meaning can be given to a^m when m is an irrational number, but this is much harder and the question is best avoided at this stage.

The meaning of a negative, zero or rational index is summarised as follows. For a positive and m, n positive integers,

$$\begin{aligned} a^0 &= 1 & a^{-n} &= \frac{1}{a^n} \\ a^{\frac{1}{2}} &= \sqrt{a} & a^{\frac{1}{n}} &= \sqrt[n]{a} \\ a^{\frac{m}{n}} &= \sqrt[n]{a^m} = (\sqrt[n]{a})^m. \end{aligned}$$

Example

Simplify

1 a $3x^3y^6 \times 5x^4yz^3$ b $64a^4b^7c \div 4a^3b^2c$

2 a $27^{\frac{4}{3}}$ b $17^0 \times 25^{-\frac{1}{2}}$.

Solution

1 a $3x^3y^6 \times 5x^4yz^3 = 15x^7y^7z^3$ b $64a^4b^7c \div 4a^3b^2c = 16ab^5$

2 a $27^{\frac{4}{3}} = (\sqrt[3]{27})^4 = 81$ b $17^0 \times 25^{-\frac{1}{2}} = \frac{1}{5}$.

Exercise 1

Simplify

$$\frac{x^{-1} + y^{-1}}{(xy)^{-1}}.$$

Expanding

Brackets

Collections of terms may be grouped together in algebra using brackets. Such a collection may then be multiplied by another such term or collection of terms. To simplify such an expression may involve expanding the brackets using the distributive law.

Example

Simplify

$$2x^2y(3x - 5y).$$

Solution

$$2x^2y(3x - 5y) = 6x^3y - 10x^2y^2.$$

Note that we used the index laws.

Binomials

We can also expand pairs of brackets. To expand the expression $(a + b)(x + y)$, we must multiply every term in the first bracket by every term in the second. Hence,

$$\begin{aligned}(a + b)(x + y) &= a(x + y) + b(x + y) \\ &= ax + ay + bx + by.\end{aligned}$$

Expressions such as this, involving a pair of brackets each containing two terms, are sometimes referred to as **binomials**. With practice, the middle step above can be left out.

Example

Expand and simplify

$$(2x + 3)(5x - 2).$$

Solution

$$\begin{aligned}(2x + 3)(5x - 2) &= 2x(5x - 2) + 3(5x - 2) \\ &= 10x^2 - 4x + 15x - 6 \\ &= 10x^2 + 11x - 6.\end{aligned}$$

Note that the first expansion can be left out.

The process can be extended to brackets containing more than two terms.

Example

Expand and simplify

$$(x - 2)(x^2 + 2x + 4).$$

Solution

$$\begin{aligned}(x - 2)(x^2 + 2x + 4) &= x(x^2 + 2x + 4) - 2(x^2 + 2x + 4) \\ &= x^3 + 2x^2 + 4x - 2x^2 - 4x - 8 \\ &= x^3 - 8.\end{aligned}$$

Exercise 2

Expand and simplify

$$(x - 1)(x^{n-1} + x^{n-2} + \cdots + x^2 + x + 1).$$

Special expansions

There are three special binomial products that arise frequently and their patterns should be learned off by heart, rather than expanding them step by step as we did above.

$$(x + y)^2 = x^2 + 2xy + y^2$$

$$(x - y)^2 = x^2 - 2xy + y^2$$

$$(x + y)(x - y) = x^2 - y^2$$

The first two of these identities are often referred to as **perfect squares**, while the third is called the **difference of squares**.

Example

Expand

$$1 \quad (x - 2)(x + 2) \qquad 2 \quad (x + 5)^2 \qquad 3 \quad (2x - 3)^2.$$

Solution

$$1 \quad (x - 2)(x + 2) = x^2 - 4 \qquad 2 \quad (x + 5)^2 = x^2 + 10x + 25$$

$$3 \quad (2x - 3)^2 = 4x^2 - 12x + 9.$$

Exercise 3

By expanding both sides (separately) prove that

$$(a + b + c)(ab + bc + ca) - abc = (a + b)(b + c)(c + a).$$

Factoring

Factoring is the reverse process to expanding. Writing the number 3128 as $8 \times 17 \times 23$ gives us useful information about the number. For example, it tells us that 3128 is divisible by 8 and 17 and 23. It also allows us to calculate $3128 \div 23$ without using a calculator. Similarly, in algebra, factoring an algebraic expression is a very useful and important skill. Indeed, it is very unwise in algebra to expand a complicated collection of brackets unless you have no other option. Factoring is often much the better thing to do. Factoring also enables us to solve certain types of equations and will reappear when we look at quadratic equations and again in the module *Polynomials*.

Basic factoring

Students should always be on the lookout for simple common factors.

For example, the terms in the expression $3x^2 + 12x$ have a common factor of $3x$ which can be factored out. Thus, $3x^2 + 12x = 3x(x + 4)$.

Sometimes after doing this we can spot a further common factor. For example,

$$\begin{aligned} 12xy - 15x - 16y + 20 &= 3x(4y - 5) - 4(4y - 5) \\ &= (3x - 4)(4y - 5). \end{aligned}$$

Exercise 4

Factorise

$$4x^3 - 12x^2 - x + 3.$$

Hence state the values of x which make this expression equal to 0.

Quadratics

A quadratic expression has the form $ax^2 + bx + c$, where a, b, c are given numbers with $a \neq 0$. In some cases a quadratic expression can be factored. For example, starting with

the expression $x^2 + 5x + 6$, we can, rather cleverly, write this as

$$\begin{aligned}x^2 + 2x + 3x + 6 &= x(x + 2) + 3(x + 2) \\ &= (x + 3)(x + 2).\end{aligned}$$

Now how did we know to split the middle term in this way? The answer lies in looking at what happens when we expand:

$$(x + a)(x + b) = x^2 + (a + b)x + ab.$$

Clearly the coefficient of x is the sum of a and b and the constant term is their product. This is why we split the $5x$ term into $2x + 3x$. In practice, when factoring $x^2 + 5x + 6$, we seek two integers whose sum is 5 and whose product is 6. Clearly the numbers are 2 and 3, and so we obtain the desired factorisation.

Example

Factor

$$x^2 + 23x - 50.$$

Solution

We seek two numbers whose sum is 23 and whose product is -50 . The numbers 25 and -2 will do. Hence,

$$x^2 + 23x - 50 = (x + 25)(x - 2).$$

Provided we can find the appropriate numbers, this method works well for **monic quadratics**, that is, for quadratics whose leading coefficient is 1.

In the case of a **non-monic quadratic**, the following algorithm is employed. It is best seen by example.

To factor $3x^2 + 14x - 5$, we first multiply the 3 and the -5 to obtain -15 . Now we find two numbers whose sum is 14 and whose product is -15 . Clearly, these are 15 and -1 . We use these numbers to split the middle term and write

$$3x^2 + 14x - 5 = 3x^2 + 15x - x - 5.$$

Now we factor in pairs, as above. Thus

$$\begin{aligned}3x^2 + 14x - 5 &= 3x^2 + 15x - x - 5 \\ &= 3x(x + 5) - 1(x + 5) \\ &= (3x - 1)(x + 5).\end{aligned}$$

Of course there are quadratic expressions such as $x^2 + x + 1$ which cannot be factored in this way.

Special factorisations

The following three special factorisations are exactly the same as the three special expansions discussed in *Expanding (Special expansions)* in this module, but used in reverse:

$$x^2 + 2xy + y^2 = (x + y)^2$$

$$x^2 - 2xy + y^2 = (x - y)^2$$

$$x^2 - y^2 = (x + y)(x - y).$$

Of these the last is probably the most important, and students need to spot this factorisation whenever it occurs.

Example

Factor

$$9x^2 - 25y^2.$$

Solution

$$9x^2 - 25y^2 = (3x + 5y)(3x - 5y).$$

Exercise 5

Factorise fully

$$(a^2 + ab)^2 - (ab + b^2)^2.$$

Algebraic fractions

An **algebraic fraction** is a fraction containing pronumerals. Later, when functions are introduced, an algebraic fraction in which both the numerator and the denominator are polynomials will be referred to as a **rational function**.

Algebraic fractions are manipulated and simplified using exactly the same rules that apply to ordinary fractions. In particular, to add or subtract them we need to find (lowest) common denominators. In many instances factoring needs to be done first so that these common factors become apparent.

For example, to simplify

$$\frac{4}{x^2 + x} - \frac{3}{x^2 - 1}$$

we need to factor each denominator using the techniques outlined above. Thus we write

$$\frac{4}{x^2 + x} - \frac{3}{x^2 - 1} = \frac{4}{x(x+1)} - \frac{3}{(x-1)(x+1)}.$$

It is now apparent that the lowest common denominator is $x(x+1)(x-1)$. Hence,

$$\begin{aligned} \frac{4}{x(x+1)} - \frac{3}{(x-1)(x+1)} &= \frac{4(x-1) - 3x}{x(x+1)(x-1)} \\ &= \frac{x-4}{x(x^2-1)}. \end{aligned}$$

It is worth mentioning in passing that we assume in all these steps that x does not take any of the values 0, 1 or -1 , since that would make the denominators equal to zero. While being aware of this, we generally do not need to make a big issue of it. It is tacitly assumed in all our calculations.

Factoring may also be very useful when multiplying two algebraic fractions, since we can then see any cancellation that may occur.

For example, to multiply

$$\frac{3x}{x^2 - 9} \times \frac{x+3}{3x^2 - 6x}$$

we factor, where possible, first:

$$\frac{3x}{x^2 - 9} \times \frac{x+3}{3x^2 - 6x} = \frac{3x}{(x-3)(x+3)} \times \frac{(x+3)}{3x(x-2)}.$$

Cancelling out the factors $3x$ and $x+3$, we have

$$\frac{3x}{(x-3)(x+3)} \times \frac{(x+3)}{3x(x-2)} = \frac{1}{(x-3)(x-2)}.$$

Exercise 6

Simplify

$$\frac{8x}{x^2 + 5x + 6} - \frac{5x}{x^2 + 3x + 2} - \frac{3x}{x^2 + 4x + 3}.$$

Linear equations and inequalities

Many problems that arise in the applications of mathematics lead naturally to equations. Equations in which the unknown only occurs to the first power are called **linear equations**. Thus

$$2x - 4 = 6 \quad \text{and} \quad \frac{5x}{2} - 7 = 4x + \frac{1}{2}$$

are examples of linear equations, whereas

$$x^2 - 5x + 6 = 0 \quad \text{and} \quad \frac{2x}{x-1} = \frac{x-3}{2x+5}$$

are not linear, but are examples of quadratic equations which will be dealt with later. Equations such as

$$x^3 - 3x^2 + 6x - 4 = 0$$

are called **polynomial equations**, and equations such as

$$e^x = x - 3$$

are often referred to as **transcendental equations**. With some exceptions, such transcendental equations are generally not dealt with in secondary school mathematics.

It is important to distinguish clearly between identities and equations. An **identity** is a statement that is true for (almost) all values of the pronumeral. For example,

$$(x + 1)^2 = x^2 + 2x + 1$$

is an identity, since when any real number is substituted for x a true statement results. An **equation** is generally only true for certain values of the pronumeral. Thus, $3x - 2 = 10$ only yields a true statement when x takes the value 4. This is called the **solution** of the equation.

Linear equations only have (at most) one solution. Can you write down a linear equation with no solution?

Linear equations

The various types of linear equations and the various strategies to solve them are dealt with at length in the module *Linear equations*, and so we will only quickly revise some of these ideas here via two examples. The basic rule throughout is that whatever you do to one side of the equation you must also do to the other.

Example

Solve

$$\frac{5x-3}{2} = 6.$$

Solution

$$\frac{5x-3}{2} = 6$$

$$5x-3 = 12$$

$$5x = 15$$

$$x = 3.$$

Example

Solve

$$\frac{2x+1}{4} = \frac{3x-1}{7}.$$

Solution

$$\frac{2x+1}{4} = \frac{3x-1}{7}$$

$$7(2x+1) = 4(3x-1)$$

$$14x+7 = 12x-4$$

$$2x = -11$$

$$x = -\frac{11}{2}.$$

Linear inequalities

A linear inequality resembles in form an equation, but with the equal sign replaced by an inequality symbol. The solution of a linear inequality is generally a range of values, rather than one specific value. Such inequalities arise naturally in problems involving words such as ‘at least’ or ‘at most’.

To solve an inequality we use the same procedures as we used when solving linear equations, with the modification that when an inequality is multiplied or divided by a negative number, the inequality is reversed.

Example

Solve

$$\frac{1-2x}{3} \leq 9.$$

Solution

$$\frac{1-2x}{3} \leq 9$$

$$1-2x \leq 27$$

$$-2x \leq 26$$

$$x \geq -13.$$

Inequalities also arise when we examine the domain of certain functions. Hence it is important that students are familiar with these before studying functions.

Quadratic equations

The general form of a quadratic equation is $ax^2 + bx + c = 0$, where a, b, c are given numbers with $a \neq 0$. These also arise naturally when solving real-world problems. For example, a ball thrown at an angle under gravity moves along a path that is parabolic. If we ask how far away the ball will land, then a quadratic equation arises.

There are a number of methods for solving quadratic equations. If the coefficients are all integers, the easiest and most common method involves factoring. But, as we have seen, it is not always possible or easy to factor a quadratic. So sometimes other techniques, such as completing the square or applying the quadratic formula, will need to be used. Students need to be comfortable and familiar with all these methods and be able to apply them strategically as needed.

When faced with a quadratic equation, we always put the equation into the standard form $ax^2 + bx + c = 0$.

Factoring method

The factoring method relies on the basic fact that if the product of two numbers is zero, then at least one of these numbers must be zero. That is, if $AB = 0$, then $A = 0$ or $B = 0$. Hence, if we have a quadratic equation in which the quadratic has been factored, then we can equate each factor to 0 and solve the resulting linear equation. The method relies on being able to factor the quadratic easily, which is not always possible.

Example

Solve

$$x^2 + 2x - 15 = 0.$$

Solution

$$x^2 + 2x - 15 = 0$$

$$(x + 5)(x - 3) = 0$$

$$x + 5 = 0 \text{ or } x - 3 = 0$$

$$x = -5 \text{ or } x = 3.$$

The same procedure works for non-monic quadratics.

We may be given an equation which is a quadratic in disguise, that is, we can rearrange it into standard form to produce a quadratic or we can make a simple substitution to turn it into a quadratic.

Example

Solve

$$\frac{5x + 7}{x - 1} = 3x + 2.$$

Solution

$$\frac{5x + 7}{x - 1} = 3x + 2$$

$$5x + 7 = (x - 1)(3x + 2)$$

$$5x + 7 = 3x^2 - x - 2$$

$$3x^2 - 6x - 9 = 0$$

$$x^2 - 2x - 3 = 0$$

$$(x - 3)(x + 1) = 0$$

$$x = 3 \text{ or } x = -1.$$

Exercise 7

Solve

$$4^x + 2^{x+2} - 32 = 0.$$

Completing the square

In the event that the quadratic does not factor nicely, there may be irrational solutions or there may be no solutions. The method of completing the square will deal with both cases. It will, of course, also work even if the quadratic factors nicely. A detailed exposition of the method is in the module *Quadratic equations* (Years 9–10).

To complete the square on the equation $x^2 - 4x - 6 = 0$, we take the -6 to the other side to produce $x^2 - 4x = 6$. Now try to write the left-hand side as a perfect square by halving the coefficient of x . The square $(x - 2)^2$ when expanded gives $x^2 - 4x + 4$, so we add 4 to both sides of the original equation to produce $(x - 2)^2 = 6 + 4 = 10$. We can now take the positive and negative square root and solve. Here is the full setting out:

$$x^2 - 4x - 6 = 0$$

$$x^2 - 4x = 6$$

$$(x - 2)^2 = 6 + 4 = 10$$

$$x - 2 = \sqrt{10}, \quad x - 2 = -\sqrt{10}$$

$$x = 2 + \sqrt{10}, \quad x = 2 - \sqrt{10}.$$

The equation has two irrational solutions.

In the case of a quadratic with no real solutions, the method will lead to a square on the left-hand side and a negative number on the right. Thus, we cannot take the square root since all (real) squares are positive or zero. What will happen in the case when the quadratic has only one solution?

Example

Solve

$$x^2 + 2x + 7 = 0.$$

Solution

$$x^2 + 2x + 7 = 0$$

$$x^2 + 2x = -7$$

$$(x + 1)^2 = -7 + 1 = -6.$$

Since the right-hand side is negative there is no solution.

If the quadratic is not monic, then you can divide both sides by the coefficient of x^2 . This will produce fractions and in this case it is often better to resort to the quadratic formula.

With practice, the method of completing the square is often quicker than using the quadratic formula and the technique itself has many other applications in mathematics. For example, it is used to find the centre and radius of a circle whose equation is not given in standard form. It is also used in integration, and in higher mathematics to deal with partial fractions and Laplace transforms.

Exercise 8

The general equation of a circle with centre (a, b) and radius r is $(x - a)^2 + (y - b)^2 = r^2$. By completing the square on the terms with x and then the terms with y , find the centre and radius of the circle

$$x^2 - 2x + y^2 - 6y = 1.$$

The quadratic formula

The quadratic formula is simply a symbolic way of completing the square. That is, it gives a formula for the end result of applying the method of completing the square to a general quadratic equation $ax^2 + bx + c = 0$. To derive the formula, it is convenient to multiply the equation by $4a$ and do a slight variation on the completing the square method.

$$ax^2 + bx + c = 0$$

$$4a^2x^2 + 4abx = -4ac$$

$$(2ax + b)^2 = b^2 - 4ac$$

$$2ax + b = \sqrt{b^2 - 4ac} \quad \text{or} \quad 2ax + b = -\sqrt{b^2 - 4ac}$$

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

If the quantity $b^2 - 4ac$ is negative, then there are no solutions. This number is called the **discriminant** and often denoted by the Greek letter Δ (pronounced 'delta'). When solving quadratic equations it is generally a good idea to calculate the discriminant Δ first and think of the quadratic formula as

$$x = \frac{-b + \sqrt{\Delta}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{\Delta}}{2a}.$$

What happens if $\Delta = 0$?

Example

Solve

$$5x^2 + 3x - 1 = 0.$$

Solution

This equation does not appear to be amenable to factoring and although completing the square would work, it looks time consuming, since the quadratic is not monic. Hence we apply the quadratic formula.

The equation is $5x^2 + 3x - 1 = 0$. Here $a = 5$, $b = 3$, $c = -1$ and so $\Delta = 3^2 - 4 \times 5 \times -1 = 29$. Hence, the solutions are

$$x = \frac{-3 + \sqrt{29}}{10} \quad \text{or} \quad x = \frac{-3 - \sqrt{29}}{10}.$$

Simultaneous linear equations

Problems in which there are two (or more) unknowns linked via linear equations often lead to simultaneous linear equations. There is a general method for solving systems of linear equations with more than two unknowns, but this will not be discussed here. In this section we will only consider systems of the form

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2,$$

where $a_1, a_2, b_1, b_2, c_1, c_2$ are given numbers and we seek to find the values of x and y .

There are two basic algebraic approaches to this problem. Students need to be familiar and comfortable with both methods.

The substitution method

Example

Solve

$$2x + y = 1$$

$$x - y = -4.$$

Solution

Number the equations:

$$2x + y = 1 \quad (1)$$

$$x - y = -4. \quad (2)$$

Make y the subject of equation (1), so $y = 1 - 2x$. Now substitute this expression for y into equation (2):

$$x - (1 - 2x) = -4$$

$$3x - 1 = -4$$

$$x = -1.$$

We can now find the value of y by substituting into the expression for y :

$$y = 1 - (2 \times -1) = 3.$$

Hence, the solution is $x = -1$, $y = 3$.

It is always wise to check that the solution actually satisfies both of the original equations.

This method is often extended to deal with simultaneous equations which are not linear.

Exercise 9

Solve simultaneously

$$y = x^2$$

$$y - x = 2.$$

The elimination method

This method is often quicker and forms the basis of the more general method for dealing with more than two unknowns.

As the name suggests, this method involves eliminating one of the unknowns by adding or subtracting one equation from a multiple of the other.

Example

Use the elimination method to solve

$$3x + 2y = 7$$

$$5x + y = 7.$$

Solution

Number the equations:

$$3x + 2y = 7 \quad (1)$$

$$5x + y = 7. \quad (2)$$

It is easiest here to eliminate y . We do this by multiplying equation (2) by 2. Number the resulting equation:

$$10x + 2y = 14. \quad (3)$$

Now subtract equation (1) from equation (3) to produce

$$7x = 7$$

$$x = 1.$$

We can now find the value of y by substituting into equation (1), thus

$$3 + 2y = 7$$

$$y = 2.$$

Hence, the solution is $x = 1$, $y = 2$.

Again, we should check that this solution satisfies the two original equations.

Formulas

A formula is a way of relating different quantities using algebra. The well-known formula

$$A = \pi r^2$$

relates the area A of a circle to its radius r . The variable A is the subject of the formula since it stands alone on the left-hand side of the equation.

It is often the case in mathematics that we need to rearrange a formula and make one of the other variables to be the subject.

For example, the cosine rule states that if a, b, c are the side lengths of a triangle and A is the angle opposite a , then

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

If we wish to find the angle in a triangle, given the three side lengths, then we need to make $\cos A$ the subject. Rearranging the terms, we have

$$2bc \cos A = b^2 + c^2 - a^2 \implies \cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

Students should be well-practised in this sort of procedure.

Exercise 10

The compound interest formula states that $A = P(1 + R)^n$, where P is the principal, R the interest rate per unit time, n the number of units of time, and A the amount to which the principal has increased. Make n the subject of the formula.

History and applications

The history of algebra was touched on in the modules listed in the *Motivation* section. It is worth remembering that our modern algebraic notation is relatively recent. Here are some examples of the different notations used from the Middle Ages onwards.

Examples of historical algebraic notation

Chuquet (1484)	$R)^2.18\bar{m}.R)^2 150$ i.e., $\sqrt{18 - \sqrt{150}}$
Pacioli (1494)	Trouame.1.n ⁰ .che giōto al suo q̄drat ⁰ faccia.12 i.e., $x + x^2 = 12$
Vander Hoecke (1514)	4 Se. -51 Pri. -30 N. dit is ghelijc $45\frac{3}{5}$ i.e., $4x^4 - 51x - 30 = 45\frac{3}{5}$
Cardano (1545)	cub ⁹ p: 6 reb ⁹ aeq̄lis 20 i.e., $x^3 + 6x = 20$
Vieta (c. 1590)	1 Q C -15 Q Q + 85 C -225 Q +274 N aequator 120 i.e., $x^6 - 15x^4 + 85x^3 - 225x^2 + 274x = 120$
Harriot (1631)	$aaa - 3.bba ===== +2.ccc$ i.e., $a^3 - 3b^2a = 2c^3$

It is obvious how much of an improvement our modern notation is, and how difficult it would be to perform the many algebraic operations discussed above without the relative ease of our modern system.

Answers to exercises

Exercise 1

$$x + y$$

Exercise 2

$$x^n - 1$$

Exercise 3

Both sides equal $a^2b + a^2c + ab^2 + 2abc + b^2c + ac^2 + bc^2$.

Exercise 4

$$(4x^2 - 1)(x - 3); x = \pm\frac{1}{2}, x = 3$$

Exercise 5

$$(a - b)(a + b)^3$$

Exercise 6

$$\frac{-13x}{(x + 1)(x + 2)(x + 3)}$$

Exercise 7

$$x = 2$$

Exercise 8

$$(x - 1)^2 + (y - 3)^2 = 11, \text{ centre } (1, 3), \text{ radius } \sqrt{11}.$$

Exercise 9

$$x = 2, y = 4 \text{ or } x = -1, y = 1.$$

Exercise 10

$$n = \frac{\log\left(\frac{A}{P}\right)}{\log(1 + R)}$$

0

1

2

3

4

5

6

7

8

9

10

11

12