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A guide for teachers – Years 11 and 12

Braking distance

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Maths delivers!
Braking distance

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Braking distance

Introduction

These notes accompany the video *Braking distance*.

Imagine that a car is travelling on a straight road. The driver sees a problem on the road ahead and so brakes suddenly to stop. The **stopping distance** is the distance that the car travels from the moment that the brakes are applied to the moment that the car stops. This is also called the **braking distance**.

If the car is initially travelling at u m/s, then the stopping distance d m travelled by the car is given by

$$d = \frac{u^2}{20}.$$

We will see later in these notes how this formula is obtained.

This formula means that the stopping distance is directly proportional to the square of the speed of the car at the instant the brakes are applied. That is,

$$d \propto u^2.$$

We refer the reader to the TIMES module *Proportion* (Years 9–10). So in general, the stopping distance d from initial speed u is given by a formula of the form

$$d = ku^2,$$

where the constant of proportionality k depends on the units being used.

What happens if the initial speed of the car is increased by 10%?

This doesn't seem much — for example, increasing your speed from 50 km/h to 55 km/h is a 10% increase. To see the effect, we take $u = 1$ and so $d = k$. A 10% increase gives $u = 1.1$, and so

$$d = (1.1)^2 k = 1.21\,k.$$

Similarly, if we increase the speed by 20% (for example from 50 km/h to 60 km/h), then

$$d = (1.2)^2 k = 1.44\, k.$$

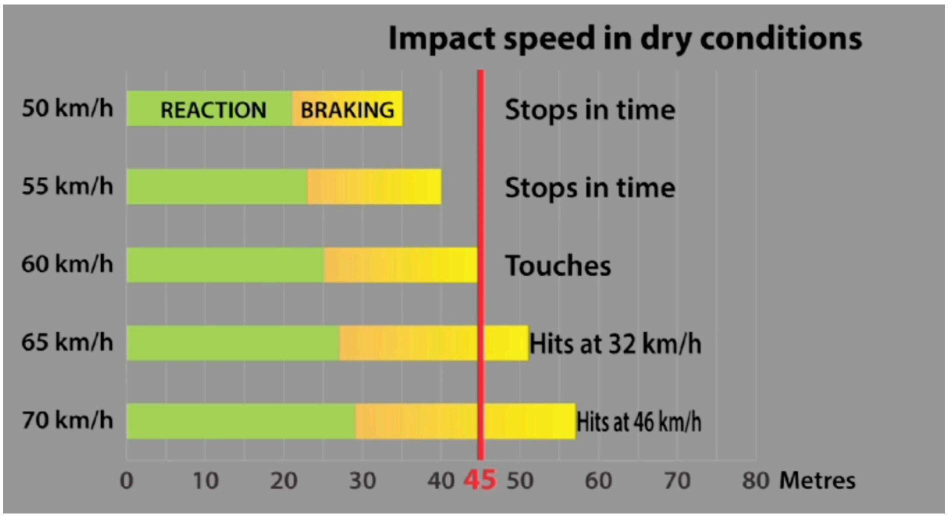
In summary:

- A 10% increase in the speed causes a 21% increase in the stopping distance.
- A 20% increase in the speed causes a 44% increase in the stopping distance.
- An a% increase in the speed causes a $\left(2a + \dfrac{a^2}{100}\right)$% increase in the stopping distance.

Exercise 1

Show that an a% increase in the speed causes a $\left(2a + \dfrac{a^2}{100}\right)$% increase in the stopping distance.

The increase in stopping distance is only one aspect of the problem with apparently small increases in speed, as illustrated by the following diagram. In these notes, we will investigate various aspects of braking distance.



Graph showing impact speeds at 45 metres.

The equations of motion for constant acceleration

We begin with the equations of motion for constant acceleration. In this section, we will assume a consistent set of units. To start with, this will be the **mks system** (metres, kilograms, seconds). Common usage will force us to depart from this later in the notes.

The five equations of motion

These equations are for a particle moving in a straight line with a constant acceleration. The history of these equations is not absolutely clear, but we do have some knowledge.

In the 14th century, scholastic philosophers at Merton College, Oxford, studied motion with constant acceleration and deduced what is now known as the *Merton rule*:

> An object with constant acceleration travels the same distance as it would have if it had constant velocity equal to the average of its initial and final velocities.

In the 17th century, Galileo Galilei (1564–1642) and others discovered that, in a void, all falling objects have the same constant acceleration. Newton (1642–1727) and Leibniz (1646–1716) built on these ideas in developing the concept of the derivative.

In the following:

- u is the initial velocity of the particle
- v is the final velocity of the particle
- a is the constant acceleration of the particle
- t is the time of motion
- x is the distance travelled.

The equations are as follows:

1. $v = u + at$
2. $x = ut + \frac{1}{2}at^2$
3. $x = \dfrac{(u+v)t}{2}$
4. $v^2 = u^2 + 2ax$
5. $x = vt - \frac{1}{2}at^2$.

Note. Each of the five equations involve four of the five variables u, v, a, t, x. If the values of three of the variables are known, then the remaining values can be found by using two of the equations.

We can use calculus to help derive these five equations.

The first equation of motion

We assume that $\ddot{x}(t) = a$, where a is a constant, and that $x(0) = 0$ and $\dot{x}(0) = u$. Integrating $\ddot{x}(t) = a$ with respect to t for the first time gives

$$v(t) = \dot{x}(t) = at + c_1.$$

Since $v(0) = \dot{x}(0) = u$, we have $c_1 = u$. Hence, we obtain the first equation of motion:

$$v(t) = u + at.$$

The second equation of motion

Integrating again gives

$$x(t) = ut + \frac{1}{2}at^2 + c_2.$$

Since $x(0) = 0$, we have $c_2 = 0$. This yields the second equation of motion:

$$x(t) = ut + \frac{1}{2}at^2.$$

The third equation of motion

We start with the second equation of motion:

$$x(t) = ut + \frac{1}{2}at^2 = \frac{2ut + at^2}{2} = \frac{ut + t(u + at)}{2}.$$

The first equation of motion is $v = u + at$. Substituting, we have

$$x(t) = \frac{ut + vt}{2} = \frac{(u + v)t}{2}.$$

The fourth equation of motion

From the first equation, we have $t = \dfrac{v - u}{a}$. Substituting this into the third equation gives

$$\begin{aligned} x &= \frac{(u + v)t}{2} \\ &= \frac{(u + v)(v - u)}{2a} \\ &= \frac{v^2 - u^2}{2a}. \end{aligned}$$

Rearranging to make v^2 the subject produces the fourth equation: $v^2 = u^2 + 2ax$.

The fifth equation of motion

From the first equation, we have $u = v - at$. Using the third equation, we obtain

$$\begin{aligned} x &= \frac{(u+v)t}{2} \\ &= \frac{(v-at+v)t}{2} \\ &= \frac{2vt-at^2}{2} \\ &= vt - \frac{1}{2}at^2, \end{aligned}$$

which is the fifth equation.

In these notes, we talk about the speed of the vehicle rather than the velocity. The speed is the magnitude of the velocity. We will always be considering positive velocity, and so the terms can be used interchangeably. However, when you apply the brakes, the force exerted by the brakes is acting in the opposite direction to the motion, and hence so is the acceleration.

Application to stopping distance

When we are discussing the motion of a car after the brakes are applied, we are interested in two questions in particular:

1. How far in metres does the car go before stopping?
2. What is the speed of the car in km/h after the car has travelled a given number of metres?

The second question is relevant for finding the speed that you would hit an object after braking a given distance from it.

It is clear from this discussion that we will use the fourth equation of motion, since this is the equation that connects speed and distance travelled:

$$v^2 = u^2 + 2ax.$$

The stopping-distance formula

We now consider the first question: *How far in metres does the car go before stopping?*

We use the fourth equation of motion, $v^2 = u^2 + 2ax$. When the car stops, we have $v = 0$, and so

$$0 = u^2 + 2ax.$$

Solving for x, we have

$$x = \frac{-u^2}{2a}.$$

We are still working with the mks system. In this situation that means we are working with metres and seconds. The corresponding units for velocity (speed) will be metres per second (m/s) and for acceleration m/s^2.

The acceleration is in the opposite direction to the motion, and so the acceleration is negative. The following formula only relates the magnitudes of the distance travelled, initial velocity and acceleration. This is a slight misuse of notation, but it is unambiguous for our purpose.

The simple formula for the **stopping distance** d is

$$d = \frac{u^2}{2a}.$$

In the following table, the stopping distance is obtained by measurement from four trials with four different cars. The brakes are applied when the car is travelling at 100 km/h. The deceleration is then calculated from the formula $a = \dfrac{u^2}{2d}$.

We will go through the details for the first calculation. We first note that

$$100\ \text{km/h} = \frac{100 \times 1000}{3600}\ \text{m/s} = \frac{250}{9}\ \text{m/s}.$$

For the first trial, the measured stopping distance is $d = 36.63$ m, and the speed when the brakes are applied is $u = \dfrac{250}{9} \approx 27.7778$ m/s, and so

$$a \approx \frac{27.7778^2}{2 \times 36.63} \approx 10.53\ \text{m/s}^2.$$

Three similar calculations give us the following table.

Measured stopping distances from 100 km/h

Car type	Stopping distance (m)	Deceleration (m/s^2)
Type A	36.63	10.53
Type B	37.95	10.17
Type C	39.86	9.68
Type D	43.56	8.86

Exercise 2

Verify the values given in the table for the deceleration of the cars of type B and C.

You can see from the table that quite a sensible choice of value for a is 10 m/s^2. In fact, a constant deceleration of 10 m/s^2 provides a good model for this context. The formula for the stopping distance d then becomes

$$d = \frac{u^2}{20}.$$

Speed while braking

We now turn to our second question concerning the motion of a car after the brakes are applied: *What is the speed of the car in km/h after the car has travelled a given number of metres?*

We again use the fourth equation of motion, $v^2 = u^2 + 2ax$. We make v the subject by taking the square root of both sides. We are only interested in the positive root, and so

$$v = \sqrt{u^2 + 2ax}.$$

When we take $a = -10\ m/s^2$, we have

$$v = \sqrt{u^2 - 20x}.$$

Rate of change of speed

The rate of change of v with respect to time is the acceleration, which is $-10\ m/s^2$ in our case. We are interested in the rate of change of the velocity with respect to distance travelled:

$$\frac{dv}{dx} = -\frac{20}{2\sqrt{u^2 - 20x}} = -\frac{10}{v}.$$

This tells us that the rate of decrease of v with respect to x is inversely proportional to the speed v. That is,

$$\text{Rate of decrease of speed with respect to distance} \propto \frac{1}{\text{Speed}}.$$

Braking from 100 km/h

When talking about braking in everyday life, we talk about speed measured in kilometres per hour and distance travelled in metres. Initially we will stay in the mks system and carry out conversions when appropriate.

For a car braking abruptly from an initial speed of 100 km/h, we take $a = -10$ m/s^2 and $u = 100$ km/h $= \frac{250}{9}$ m/s. We take the positive square root, and so

$$v = \sqrt{\left(\frac{250}{9}\right)^2 - 20x} = \frac{1}{9}\sqrt{250^2 - 1620x}.$$

In order to get a feel for this formula, we ask the following question.

Example

Use the formula derived above to find the braking distance of a car travelling at 100 km/h.

Solution

When $v = 0$, we have

$$0 = \frac{1}{9}\sqrt{250^2 - 1620x}.$$

Solving for x, we find that the braking distance is

$$x = \frac{250^2}{1620} = \frac{3125}{81} \approx 38.58 \text{ metres}.$$

This is as expected. We now look at the speed of the car at various distances measured from where the car first brakes. We are again considering braking from a speed of 100 km/h. We use $v = \frac{1}{9}\sqrt{250^2 - 1620x}$.

For example, at a distance of $x = 5$ metres, we have

$$v = \frac{1}{9}\sqrt{250^2 - 1620 \times 5} = \frac{40\sqrt{34}}{9} \approx 25.92 \text{ m/s}.$$

We change this to kilometres per hour by multiplying by $\frac{3600}{1000}$:

$$v = \frac{40\sqrt{34}}{9} \text{ m/s} \approx 93.30 \text{ km/h}.$$

The following table gives the speed of the car at each five-metre mark.

Speed of car while braking from 100 km/h

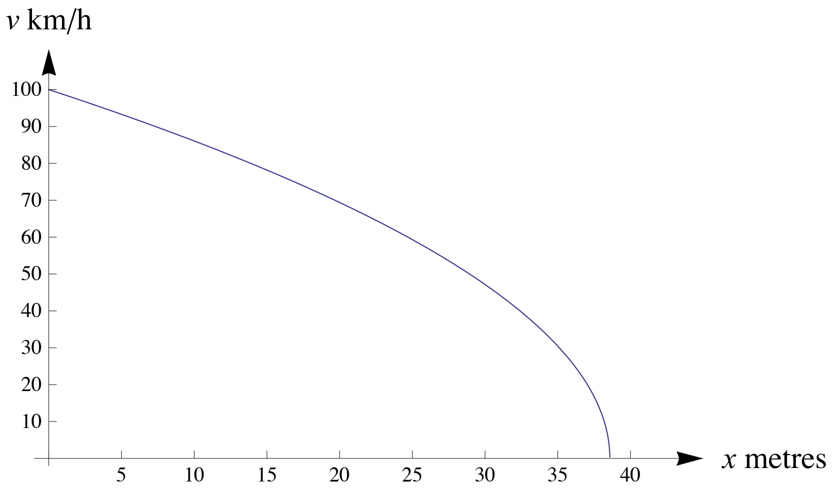
Distance travelled (m)	Speed (km/h)
0	100.00
5	93.30
10	86.07
15	78.18
20	69.40
25	59.33
30	47.16
35	30.46

We know that the car stops at 38.58 metres, but we can see from the table that the car is still going at 30.46 km/h at 35 metres.

We now give a formula for the speed of the car in kilometres per hour in terms of the distance covered in metres. This is somewhat unorthodox, but is appropriate in this context. If the brakes are first applied at a speed of 100 km/h, then the formula for the car's speed v in km/h in terms of the distance travelled x in metres is

$$v = \frac{2}{5}\sqrt{250^2 - 1620x}.$$

We plot the graph of speed v versus distance travelled x from where the brakes are first applied.



Graph showing speed versus distance travelled.

From this graph, we can see that the car slows more rapidly as it gets closer to stopping. The graph also illustrates that

$$\text{Rate of decrease of speed with respect to distance} \propto \frac{1}{\text{Speed}}.$$

We dwell on this a little longer.

Example

The brakes of a car are applied from when the car is doing 100 km/h. How far has the car gone when the speed of the car is 50 km/h?

Solution

The equation we use is

$$v = \frac{1}{9}\sqrt{250^2 - 1620x}.$$

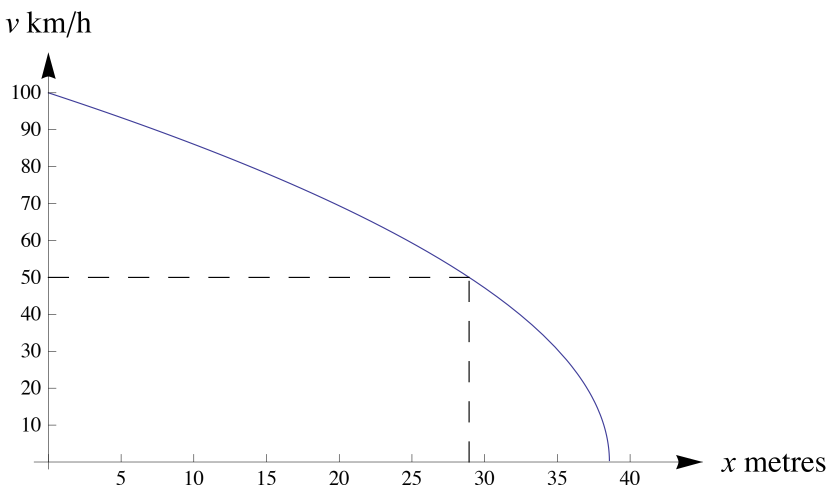
This equation gives v in m/s. We have

$$50 \text{ km/h} = 50 \times \frac{1000}{3600} \text{ m/s} = \frac{125}{9} \text{ m/s}.$$

Solving the equation

$$\frac{125}{9} = \frac{1}{9}\sqrt{250^2 - 1620x}$$

gives $x \approx 28.94$ metres. We can see this graphically as follows.



Graph showing the point where half the initial speed is reached.

Exercise 3

The brakes of a car are applied from when the car is doing 100 km/h. How far has the car gone when the speed of the car is 25 km/h?

Comparison of braking from 60 km/h and 65 km/h

Some anti-speeding campaigns have used the slogan ‘Wipe off five’. We will see why this is so important.

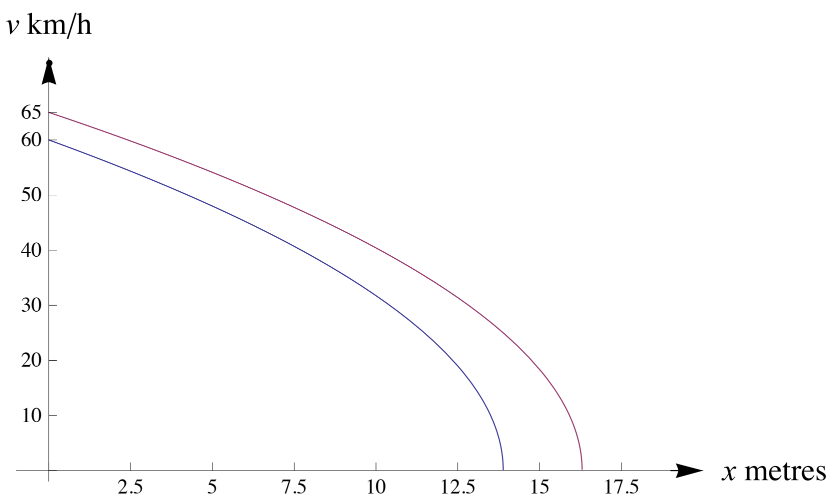
- **Initial speed 60 km/h.** If the brakes are first applied at a speed of 60 km/h, then the formula for the car’s speed v_1 (in km/h) in terms of the distance x (in m) travelled after the brakes are first applied is

$$v_1 = \frac{6}{5}\sqrt{2500 - 180x}.$$

- **Initial speed 65 km/h.** If the brakes are first applied at a speed of 65 km/h, then the formula for the car’s speed v_2 (in km/h) in terms of the distance x (in m) travelled after the brakes are first applied is

$$v_2 = \frac{1}{5}\sqrt{325^2 - 6480x}.$$

The following diagram gives the graphs of the two functions v_1 and v_2 defined above.



Graphs for initial speeds 60 km/h and 65 km/h.

In the following table, we compare the two situations.

Speed of car while braking from 60 km/h and 65 km/h

Distance travelled	Speed	
0 m	**60 km/h**	**65 km/h**
5 m	48 km/h	54 km/h
10 m	32 km/h	40 km/h
12 m	22 km/h	33 km/h
13 m	15 km/h	29 km/h
14 m	stationary	24 km/h
15 m	stationary	18 km/h

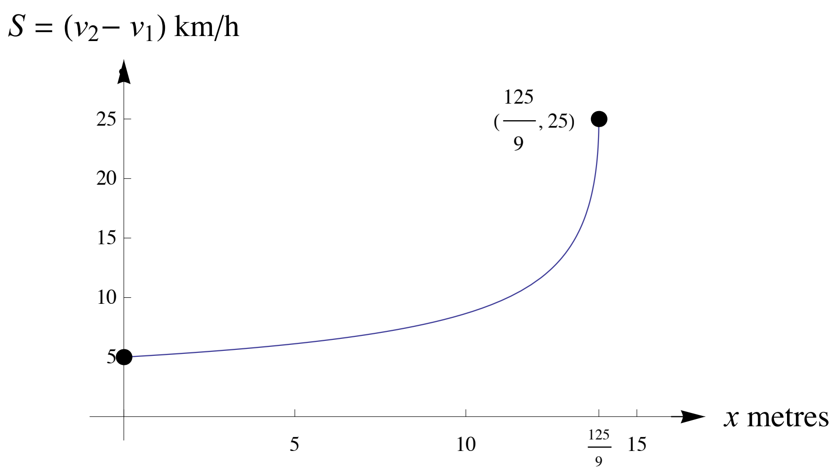
To emphasise the difference in speeds while braking for these two different initial speeds, we consider the difference $v_2 - v_1$. We have

$$S = v_2 - v_1 = \frac{1}{5}\sqrt{325^2 - 6480x} - \frac{6}{5}\sqrt{2500 - 180x}.$$

It follows that

$$\frac{dS}{dx} = \frac{648}{5}\Big(\frac{1}{v_1} - \frac{1}{v_2}\Big).$$

We now consider the graph of $S = v_2 - v_1$ against x. We note that $v_1 \geq 0$ for x in the interval $\big[0, \frac{125}{9}\big]$, and this is the domain for which the difference S makes sense.



Graph of $v_2 - v_1$ versus x.

From the graph, we see that the rate of change of the difference increases as the distance travelled increases.

We also see that when the car with initial speed 60 km/h stops, the car with initial speed 65 km/h is still travelling at 25 km/h. That is, if you applied the brakes when you were $\frac{125}{9}$ metres away from an object and you were travelling at 60 km/h, you would stop just in time; but if you were travelling at 65 km/h, you would hit the object at 25 km/h.

Thinking time

The total distance travelled from the moment that the driver perceives the emergency to when the car stops consists of two parts:

1. **Thinking distance:** the distance travelled before the driver actually starts braking.
2. **Braking distance:** the distance the car then travels before coming to rest.

Thus we can write

$$\text{Total distance} = \text{Thinking distance} + \text{Braking distance}.$$

An average thinking time is 0.68 seconds. This value was found by experiment for a particular driver. The thinking time can be much longer. It depends on many factors.

The formula for the distance x m travelled by a particle with constant velocity u m/s for a time t seconds is

$$x = ut.$$

Example

A car is travelling at 100 km/h when the driver sees something that requires rapid braking. Find the total distance (in metres) travelled by the car from the moment the driver first sees the problem until the car stops.

Solution

For an initial speed of $u = 100 \text{ km/h} = \dfrac{250}{9}$ m/s, we have

$$\text{Thinking distance} = \frac{250}{9} \times 0.68 \approx 18.9 \text{ metres},$$

$$\text{Braking distance} = \frac{u^2}{20} = \frac{3125}{81} \approx 38.6 \text{ metres}.$$

Hence,

$$\text{Total distance} \approx 18.9 + 38.6 = 57.5 \text{ metres}.$$

Exercise 4

Find the 'Total distance' for a car initially travelling with a speed of

a 60 km/h

b 65 km/h.

Exercise 5

Two cars are travelling at 60 km/h along a straight road. The front of the second car is 5 metres behind the back of the first car. The first car applies its brakes. Assume the second driver has a thinking time of 0.68 seconds. Will the second car hit the first car? If so, give the time of collision and the speed of each of the cars.

Exercise 6

Sometimes a non-constant acceleration is used to model the motion of a braking car. Assume the acceleration a m/s^2 of a braking car depends on its speed v m/s as follows:

$$a = -(0.660 - 0.001\,v)\,g,$$

where g is the acceleration due to gravity. Taking $g = 9.8$ m/s^2, calculate the stopping distance of the car from

a 60 km/h

b 65 km/h

c 100 km/h.

Answers to exercises

Exercise 1

The stopping distance is proportional to the square of the initial speed. We have

$$\begin{aligned}\left(1+\frac{a}{100}\right)^2 &= 1+\frac{2a}{100}+\frac{a^2}{10\,000}\\ &= 1+\left(2a+\frac{a^2}{100}\right)\times\frac{1}{100}\\ &= 100\% + \left(2a+\frac{a^2}{100}\right)\%.\end{aligned}$$

So an $a\%$ increase in initial speed causes a $\left(2a+\frac{a^2}{100}\right)\%$ increase in stopping distance.

Exercise 2

For trial B, the measured stopping distance is $d = 37.95$ m, and the speed when the brakes are applied is $u = \frac{250}{9} \approx 27.7778$ m/s, and so

$$a \approx \frac{27.7778^2}{2\times 37.95} \approx 10.17\ \mathrm{m/s^2}.$$

For trial C, the measured stopping distance is $d = 39.86$ m, and the speed when the brakes are applied is $u = \frac{250}{9} \approx 27.7778$ m/s, and so

$$a \approx \frac{27.7778^2}{2\times 39.86} \approx 9.68\ \mathrm{m/s^2}.$$

Exercise 3

We use the equation

$$v = \frac{1}{9}\sqrt{250^2 - 1620x},$$

where the speed v is measured in m/s. We have

$$25\ \mathrm{km/h} = 25\times\frac{1000}{3600}\ \mathrm{m/s} = \frac{125}{18}\ \mathrm{m/s}.$$

Solving the equation

$$\frac{125}{18} = \frac{1}{9}\sqrt{250^2 - 1620x}$$

gives $x \approx 36.17$ metres.

Exercise 4

a Note that 60 km/h $= \frac{50}{3}$ m/s, and so

- Thinking distance $= \frac{50}{3} \times 0.68 \approx 11.3$ metres
- Braking distance $= \left(\frac{50}{3}\right)^2 \times \frac{1}{20} = \frac{125}{9} \approx 13.9$ metres
- Total distance $\approx 11.3 + 13.9 = 25.2$ metres.

b Note that 65 km/h $= \frac{325}{18}$ m/s, and so

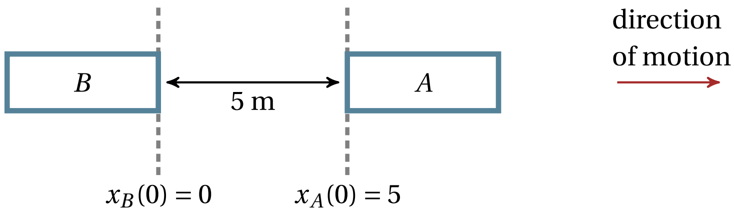
- Thinking distance $= \frac{325}{18} \times 0.68 \approx 12.3$ metres
- Braking distance $= \frac{325^2}{6480} \approx 16.3$ metres
- Total distance $\approx 12.3 + 16.3 = 28.6$ metres.

Exercise 5

Let t be the time in seconds from when the first car (A) brakes. We measure distance in metres, and take the origin to be the position of the front of the second car (B) at time $t = 0$. Relative to this origin, let

- $x_A(t)$ be the position of the back of the first car at time t
- $x_B(t)$ be the position of the front of the second car at time t.

Then the following diagram shows the situation at time $t = 0$.



Note that 60 km/h $= \frac{50}{3}$ m/s.

Car A. The velocity of the first car at time t is given by

$$\dot{x}_A(t) = \frac{50}{3} - 10t.$$

The position of the back of the first car at time t is given by

$$x_A(t) = \frac{50}{3}t - 5t^2 + 5.$$

Car B. The velocity of the second car at time t is given by

$$\dot{x}_B(t) = \begin{cases} \frac{50}{3} & \text{if } t \le 0.68, \\ \frac{50}{3} - 10(t - 0.68) & \text{if } t > 0.68. \end{cases}$$

The position of the front of the second car at time t is given by

$$x_B(t) = \begin{cases} \frac{50}{3}t & \text{if } t \le 0.68, \\ \frac{50}{3} \times 0.68 + \frac{50}{3}(t - 0.68) - 5(t - 0.68)^2 & \text{if } t > 0.68. \end{cases}$$

Collision. We solve the equation $x_A(t) = x_B(t)$ to find that the second car runs into the first car at time

$$t = \frac{1 + 0.68^2}{2 \times 0.68} \approx 1.075 \text{ seconds.}$$

This is found by solving the equation given that $t > 0.68$. It is easily shown that the collision does not take place during the second driver's thinking time. (Note also that the equations only make sense for $\dot{x}_A(t) \ge 0$, which corresponds to $0 \le t \le \frac{5}{3}$.)

At the time of the collision, the first car is travelling at approximately 21 km/h, and the second car at approximately 46 km/h.

Exercise 6

We will simplify the appearance of the equation by writing $p = 0.660$ and $q = 0.001$. The equation is

$$a = -(p - qv)g.$$

We are interested in the distance travelled and the velocity, and so we will work with

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}.$$

We have

$$v\frac{dv}{dx} = (qv - p)g,$$

and hence

$$\frac{dx}{dv} = \frac{v}{(qv - p)g}.$$

We integrate to find

$$x = \frac{v}{gq} + \frac{p \log_e |p - qv|}{gq^2} + c.$$

Let u be the original velocity of the car. Then

$$x = \frac{1}{gq}\left((v-u) + \frac{p}{q}\log_e\left|\frac{p-qv}{p-qu}\right|\right).$$

To find the stopping distance, we put $v = 0$. This implies

$$x = \frac{1}{gq}\left(-u + \frac{p}{q}\log_e\left|\frac{p}{p-qu}\right|\right).$$

a When $u = \frac{50}{3}$ m/s (60 km/h), the stopping distance is 21.84 metres.

b When $u = \frac{325}{18}$ m/s (65 km/h), the stopping distance is 25.67 metres.

c When $u = \frac{250}{9}$ m/s (100 km/h), the stopping distance is 61.38 metres.

0 1 2 3 4 5 6 7 8 9 10 11 12