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The Improving Mathematics Education in Schools (TIMES) Project

THE UNITARY METHOD

A guide for teachers - Years 7–8

NUMBER AND ALGEBRA : Module 17

June 2011

YEARS 7 8

The Unitary Method

(Number and Algebra : Module 17)

For teachers of Primary and Secondary Mathematics

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THE UNITARY METHOD

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NUMBER AND ALGEBRA
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YEARS 7 8

THE UNITARY METHOD

ASSUMED KNOWLEDGE

The unitary method provides an alternative approach to solving problems in various topics in the Years 7–8 course interrelated by ratio — fractions, percentages, rates, interest, ratio and similarity. Its appeal lies in its transparent logic, which often allows problems to be solved by mental arithmetic.

The assumed knowledge for this module depends on the topic in which the unitary method is being applied.

- Fluency in multiplication and division are always essential.
- HCF (highest common factor) and perhaps LCM (lowest common multiple) are needed once the unit is no longer 1.
- Some elementary work with the unitary method could precede fractions, but the unitary method would normally follow the standard fraction algorithms.
- Percentages and simple percentage problems would normally precede the unitary method approach to them.
- Ratio calculations are an aspect of the unitary method.

MOTIVATION

Ratio is central to arithmetic. Besides explicit work on ratio, school courses teach three standard approaches to ratio — fractions, decimals and percentages. These involve extending the whole number system to the set of rational, or fractional, numbers. The unitary method, in contrast, is able to deal with many ratio problems using only whole numbers. This has two great advantages.

First, the unitary method often allows problems to be solved mentally, in contrast to the standard written algorithms. Thus the unitary method is an important part of learning mental arithmetic, and once grasped, can be used quickly for all sorts of calculations in everyday life and in financial situations.

Secondly, fluency in the unitary method can often lead to a better understanding of the way in which a fraction is made up of two whole numbers, the numerator and the denominator.

In particular, the idea of taking the HCF of two numbers can sometimes seem rather removed from practical applications, but in the unitary method, working with the HCF is instinctive because it makes calculations easier.

CONTENT

THE BASIC UNITARY METHOD

Although the unitary method may eventually become a mental algorithm, it should be taught using carefully written sentences so that its logic is clear. Usually three successive parallel sentences are required. The first sentence restates the problem, and the last states its solution.

EXAMPLE

If 4 mangoes cost $12, how much do 9 mangoes cost?

SOLUTION

	4 mangoes cost $12
$\div 4$	1 mango costs $3
$\times 9$	9 mangoes cost $27.

This very simple example already demonstrates important aspects of the method.

1 The method relies on a sequence of parallel sentences. Once the method is mastered, the successive sentences in easier examples can merely be spoken or thought.

2 The problem can, of course, be solved in one step by multiplying by $\frac{9}{4}$:

	4 mangoes cost $12
$\times \frac{9}{4}$	9 mangoes cost $\$12 \times \frac{9}{4} = \$3 \times 9 = \$27$.

These calculations are no longer mental calculations for most people. The fraction multiplication on the second line, however, carries out exactly the same division by 4 followed by multiplication by 9 as in the original solution.

EXERCISE 1

A greengrocer is selling apples at 4 for \$6.80, and oranges at 11 for \$19.80.
Which costs more, 10 apples or 10 oranges, and by how much?

SOLVING PROBLEMS IN BOTH DIRECTIONS BY THE UNITARY METHOD

One of the benefits of the unitary method is that it is just as easy to solve problems posed in both directions as illustrated in the following example.

EXAMPLE

Twenty tiles weigh 5 kg. How much do 7 tiles weigh? If a man wants to carry no more than 12 kg, how many tiles can there be in one load?

SOLUTION

For the first question, we take the unit as '1 tile', and the calculations are determined by the numbers in the first half of the sentence.

	20 tiles weigh 5 kg
$\div 20$	1 tile weighs $\frac{1}{4}$ kg (or 250 g)
$\times 7$	7 tiles weigh $1\frac{3}{4}$ kg (or 1750 g).

For the second question, we take the unit as '1 kg', and the calculations are determined by the numbers in the second half of the sentence.

	20 tiles weigh 5 kg
$\div 5$	4 tiles weigh 1 kg
$\times 12$	48 tiles weigh 12 kg.
	So a load is no more than 48 tiles.

SOLVING FRACTION PROBLEMS BY THE UNITARY METHOD

The unitary method is often useful for multiplying and dividing by fractions, because it makes immediate sense, and can be done mentally if the numbers work out well. Because of this, it can be a helpful way to introduce or to justify the standard written algorithms for fractions, particularly division by a fraction.

EXAMPLE

Find $\frac{3}{7}$ of \$427.

SOLUTION

The whole is \$427

$\div 7$	$\frac{1}{7}$ is \$61
$\times 3$	$\frac{3}{7}$ is \$183.

EXAMPLE

A tank that is $\frac{3}{5}$ full contains 1800 litres. What is its capacity?

SOLUTION

$\frac{3}{5}$ is 1800 litres

$\frac{1}{5}$ is 600 litres

$\frac{5}{5}$ is 3000 litres.

Hence the tank's capacity is 3000 litres.

This last calculation is exactly the same as division by $\frac{3}{5}$.

$$\text{capacity} = 1800 \div \frac{3}{5} = 1800 \times \frac{5}{3} = 3000 \text{ litres.}$$

EXERCISE 2

a If $\frac{3}{8}$ of a number is x, use the unitary method to find the number.

b If $\frac{a}{b}$ of a number is x, use the unitary method to find the number. Hence produce an alternative justification of the rule, '*To divide by a fraction, multiply by its reciprocal*'.

RATES AND THE UNITARY METHOD

Rates problems are well suited to the unitary method because they make sense of the units for rates. The formula $V = \frac{D}{T}$ links distance D, time T and speed V, but with reasonably simple numbers, we can understand speed problems better using the unitary method.

EXAMPLE

A car is travelling at 40 km/h.

a How long does it take to go 7 km?

b How far does it go in 7 minutes?

c What is the speed of another car that travels 5 km in 7 minutes?

SOLUTION

a 40 km takes 60 minutes (Rewriting the speed as a sentence is the key step.)

$\div 40$ 1 km takes $1\frac{1}{2}$ minutes

$\times 7$ 7 km takes $10\frac{1}{2}$ minutes.

b 40 km takes 60 minutes (OR 'In 60 minutes the car travels 40 km.')

$\div 60$ $\frac{2}{3}$ km takes 1 minute

$\times 7$ $4\frac{2}{3}$ km takes 7 minutes.

c For the second car, 14 km takes 7 minutes

$\div 7$ 2 km takes 1 minute

$\times 60$ 120 km takes 60 minutes.

Hence the average speed is 120 km/h.

Changes of units often seem more straightforward when handled this way. Part **b** of the following exercise is a notorious problem, but the problem is easily solved by changing the units from 'minutes per bathtub' to 'bathtubs per hour', as required in part **a**.

EXERCISE 3

The bathtub problem

a A tap can fill a bath in 20 minutes. Express this in units of 'bathtubs per hour'.

b A bath has two taps. One tap takes 20 minutes to fill the bath alone, the other tap takes 40 minutes alone. How long will it take to fill to fill the bath if both taps are running together?

USING THE HCF IN THE UNITARY METHOD

Now consider the following variation of the first problem in this module.

EXAMPLE

If 12 mangoes cost $27, how much do 20 mangoes cost?

SOLUTION

12 mangoes cost $27

$\div 3$ 4 mangoes cost $9

$\times 5$ 20 mangoes cost $45.

1 In the solution presented above, the quantity '4 mangoes' has been chosen as the 'unit' of the unitary method. This is quicker and easier than choosing '1 mango' because the divisors and multiplies are smaller, and in this case it avoids fractions of a dollar.

2 The number 4 is the HCF (highest common factor) of 12 and 20, and taking the HCF is clearly a natural thing to do here. It is good to see the HCF take its proper place in mental arithmetic.

3 It is also possible to solve the problem using the LCM (lowest common multiple) of 12 and 20 instead of their HCF:

12 mangoes cost $27

$\times 5$ 60 mangoes cost $135

$\div 3$ 20 mangoes cost $45.

This is less attractive, however, because it involves larger numbers.

4 It is possible, but a distraction, to use addition and subtraction steps. For example,

12 mangoes cost $27

$\div 3$ 4 mangoes cost $9

$\times 2$ 8 mangoes cost $18.

Adding the first and third lines, 20 mangoes cost $45.

Addition and subtraction steps upset the simplicity of the method, and they also work against developing intuition about the HCF and the LCM.

EXERCISE 4

A 350 gram jar of marmalade costs \$10.50 and a 550 gram jar of the same marmalade costs \$13.20. Use the unitary method to find how much more jam costs per kilogram when bought in the small jar.

Using the HCF of the two numbers as the 'unit' is helpful when the numbers are larger, and applies just as straightforwardly to problems that seem to be posed in reverse, and to rate problems.

EXAMPLE

Wesley copied out a 6000-word essay in 80 minutes. How many words can he copy out in 6 hours?

SOLUTION

Six hours is 360 minutes, and the HCF of 360 and 80 is 40.

	6000 words take 80 minutes
$\div 2$	3000 words take 40 minutes
$\times 9$	27 000 words take 360 minutes.

Wesley can copy out 27 000 words in 6 hours.

EXAMPLE

If a car takes 24 minutes to cover a 15 km length of road, how long will it take to travel 100 km?

SOLUTION

	15 km takes 24 minutes
$\div 3$	5 km takes 8 minutes (The HCF of 15 and 100 is 5.)
$\times 20$	100 km takes 160 minutes

The car will take 2 hours and 40 minutes to travel 100 km.

EXERCISE 5

Convert 1 metre per second to kilometres per hour, and then to minutes per kilometre.

PERCENTAGE PROBLEMS AND THE UNITARY METHOD

Percentage problems can be done by the unitary method, particularly problems that involve percentage increase and decrease. The method provides a particularly clear approach to solving reverse percentage problems, which traditionally cause confusion.

EXAMPLE

Find 30% of \$320.

SOLUTION

	100% is \$320
÷ 10	10% is \$32
× 3	30% is \$96.

In the next problem, some people prefer to take the 20% as the unit because 20 is the HCF of 100 and 40, while others prefer to take 10% as the unit because it is easy to go from 10% to 100%.

EXAMPLE

A shirt has been discounted 60% and now costs \$70. What did it cost originally?

SOLUTION

	40% is \$70 (This is the key step!)			
÷ 2	20% is \$35	OR	÷ 4	10% is \$17.50
× 5	100% is \$175.		× 10	100% is \$175.

Hence the original price was \$175.

Problems involving the GST, particularly reverse problems, are well suited to the unitary method.

EXAMPLE

A replacement car motor costs \$2860, which includes GST of 10% on the cost price. What was the cost before the GST was added?

SOLUTION

	110% of the cost before GST is \$2860 (This is the key step!)
÷ 11	10% of the cost before GST is \$260
× 10	100% of the cost before GST is \$2600.

Note that if the original price is a multiple of \$10, the price with GST added will always be a multiple of 11.

EXERCISE 6

Use the unitary method to justify the well-known GST rule: 'If GST has already been added to the price, find the original amount by dividing by 11, then multiplying by 10'.

SIMPLE INTEREST AND THE UNITARY METHOD

The simple interest formula $I = PRT$ should be understood as dealing first with the rate per unit time, then with the number of units of time. The unitary method can help to clarify this.

EXAMPLE

Find the simple interest on \$6000 for 4 years at a rate of 8% per annum.

SOLUTION

principal = \$6000

÷ 100 interest at 1% pa for 1 year = \$60

× 8 interest at 8% pa for 1 year = \$480

× 4 interest at 8% pa for 4 years = \$1920.

Doing exactly these same steps with the pronumerals P, R and T instead of the quantities \$6000, 8% and 4 years generates the standard formula I = PRT.

Reverse problems can always be done by substitution into the formula, but an initial approach using the unitary method often allows the problem to be done mentally, as well as giving a better intuitive grasp of the situation.

EXAMPLE

What interest rate will generate simple interest of \$24 000 over three years on a principal of \$160 000?

SOLUTION

interest for 3 years = \$24 000

÷ 3 interest for 1 year = \$8000

$$\text{interest rate} = \frac{8000}{160\,000} \times \frac{100}{1}\%$$

$$= 5\%.$$

The passage from simple interest to compound interest involves the understanding that when say 5% interest is added to an amount, the new amount is 105% of the old amount. Applying the unitary method to reverse problems involving the final amount thus prepares the way for compound interest calculations in Years 9–10.

EXAMPLE

What was the principal one year ago, if the final amount is now \$4444 after simple interest at 10% per annum has just been added?

SOLUTION

110% of principal = \$4444 (Again, this is the key step!)

$\div 11$ 10% of principal = \$404

$\times 10$ 100% of principal = \$4040.

RATIOS AND THE UNITARY METHOD

The methods used in ratio problems are a variation on the same theme — the 'parts' used to solve ratio problems are the 'units' of the unitary method.

EXAMPLE

Divide \$840 amongst Alfred, Barnaby and Claude so that Alfred has five times as much as Barnaby and Claude has six times as much as Barnaby.

SOLUTION

Begin by dividing the \$840 into 5 + 1 + 6 = 12 parts.

12 parts = \$840

$\div 12$ 1 part = \$70

Multiplying this sentence by 5, and by 6, gives

5 parts = \$350 and 6 parts = \$420.

Hence Alfred receives \$350, Barnaby receives \$70, and Claude receives \$420.

GEOMETRY AND THE UNITARY METHOD

Calculations associated with the enlargement transformation and similarity can be handled by the successive sentences of the unitary method as well as by the more usual approach through the similarity factor.

EXAMPLE

A proposed building will have length 270 metres, and the architect has prepared a scale model of length 2.1 metres. What will the actual dimensions of a room be whose measurements for the model are 70 mm × 105 mm?

SOLUTION

2100 mm corresponds to 270 metres

÷ 30 — 70 mm corresponds to 9 metres

÷ 2 — 35 mm corresponds to $4\frac{1}{2}$ metres

× 3 — 105 mm corresponds to $13\frac{1}{2}$ metres:

Hence the actual room will be 9 metres × $13\frac{1}{2}$ metres.

EXERCISE 7

(This will need written or calculator assistance.) Jupiter is about 45 light-minutes from the Sun, and Proxima Centauri, our closest star, is about 4.3 light-years away. If a model of the Solar System and its environs is built in which Jupiter is 3 metres from the Sun, approximately how far away should Proxima Centauri be?

LINKS FORWARD

The principal purpose of the unitary method is to provide within the Years 7–8 course an alternative way to explain, and to link together, the various topics of ratio arithmetic — fractions, decimals, percentages, rates, simple and compound interest, enlargement and similarity — and to tie these topics in turn to mental arithmetic of the whole numbers and to the HCF.

Ratio ideas continue to be of fundamental importance in later years, when they are essential elements of gradient, of the trigonometric functions, of proportion and variation, and later of the derivative. More systematic algebraic methods are required for these topics, but an idea like gradient $\frac{2}{5}$ easily comes to life with a simple observation like

> 'On a hillside with gradient $\frac{2}{5}$, every time you walk 100 metres on the map, you climb 40 metres in height.'

This would be justified, of course, by the unitary method. Again, sin 30° = $\frac{1}{2}$ means that

> 'On a hillside with an angle of inclination of 30°, if you walk 100 metres upwards along the track, you climb 50 metres in height.'

Later, it will be important to interpret the derivative in terms of rates of various kinds, such as velocity and acceleration. At this stage, constant rates, dealt with by the unitary method and linear functions, need to be contrasted with the variable rates that calculus can deal with.

APPLICATIONS

Proficiency in ratio arithmetic is necessary in some professions. For example, nurses, doctors, veterinarians and dentists routinely administer drugs whose dosages depends on the weight of the patient. A drug is often supplied with one dilution, and they need to dilute it further before administering it. Builders have to deal constantly with the density of their wood, stone or brick as they work out safe loads. They need trigonometry whenever there are angles involved, such as in a roof, and they are usually trained to apply trigonometry using the unitary method rather than the algebraic methods of secondary school. Cooks and bakers and building suppliers usually work out their quantities using proportions.

Wherever ratio and proportions are used, the unitary method provides a clear and straightforward logic to the calculations. Using a calculator is sensible once the methods are mastered, but on the other hand the unitary method allows easy mental estimation of quantities when an exact answer is not needed. For example, exact calculations in finance are carried out on a spreadsheet or using a database, but everyone with any understanding of banking, investment and business generally uses the unitary method to estimate amounts rapidly and with reasonable accuracy.

ANSWERS TO EXERCISES

EXERCISE 1

10 oranges costs \$1 more than 10 apples.

EXERCISE 2

a $\frac{8x}{3}$ **b** $\frac{bx}{a}$

EXERCISE 3

a 3 bathtubs per hour **b** $\frac{40}{3}$ minutes

EXERCISE 4

\$6 per kg more

EXERCISE 5

3.6 km per hour

$16\frac{2}{3}$ minutes per kilometre

EXERCISE 7

150.672 kilometres away

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The modules are organised under the strand titles of the Australian Curriculum:

- Number and Algebra
- Measurement and Geometry
- Statistics and Probability

The modules are written for teachers. Each module contains a discussion of a component of the mathematics curriculum up to the end of Year 10.

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